The Concept of Phase-Space
in Quantum Mechanics
Some Simple Derivations of Old Formulae in Physics

V. Devanathan[*]

The Academy of Sciences, Chennai
Department of Nuclear Physics, University of Madras
Guindy Campus, Chennai - 600 025

Abstract: The concept of phase-space in quantum mechanics is explained and used to derive in a simple way the old formulae, such as Planck's law of radiation, the Fermi energy of electrons or nucleons in a system and Debye's formula for specific heat of solids, given in standard textbooks. The simplicity in deriving some of the old formulae will be useful to the graduate and post-graduate students in Science and Engineering.

1 Introduction

Classical Physics is based on the dichotomy of the universe - particles and waves. Separate theories have been developed for studying the particles in motion and for wave propagation. The quantum theory has unified particles and waves and is based on the wave-particle duality.

The birth of the Quantum Theory of light can be traced to the year 1900 when Planck presented his law of Black Body Radiation. He found the inadequacy of classical physics to explain the experimental findings of the spectral distribution of black body radiation at any given temperature and had to invoke an hypothesis that the radiation is absorbed or emitted in discrete quanta by the walls of the container. Einstein, in the year 1905, extended Planck's hypothesis and made a bold suggestion that light is made up of discrete quanta in order to explain the photoelectric effect.

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[*]Email: vdevanathan@hotmail.com

Louis de Broglie, in the year 1924 in his doctoral thesis, argued that if light can exhibit both wave and particle behaviour, why not particles exhibit wave behaviour? Soon, the experiments of Davisson and Germer and Thompson revealed that electrons do exhibit interference and diffraction phenomena just as waves.

We have come a along way, unifying the concept of particles and waves and accepting the particle-wave duality of both particles and waves. So, the concept of phase-space [1] is applicable to both particles and waves. But still in many textbooks [2 - 9], Planck's law of radiation, the energy distribution of Fermions in a bound system and Debye's theory of specific heat of solids are presented using the state of knowledge that existed at the time of their original creation. In this article, it is emphasized that they can be presented in a simple alternative way using the state of knowledge available today.

It is pointed out that the concept of phase-space [1] simplifies the derivation of some of the old formulae in Physics. Besides, the discussion of those formulae and consequent developments in the several subjects are also presented for the sake of completeness. Since these topics are studied in the degree and postgraduate degree courses in Science and Engineering, this review article will be beneficial to the students and faculty members in those institutions conducting these courses.

2　Concept of phase-space

The state of a particle is specified by specifying both its position $\boldsymbol{r}$ $(= x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k})$ and momentum $\boldsymbol{p}$ $(= p_x\boldsymbol{i} + p_y\boldsymbol{j} + p_z\boldsymbol{k})$, where $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ denote the unit vectors in Cartesian coordinate system. The phase-space is the product of ordinary space and momentum space. It is a six-dimensional space in which the state of a particle can be represented.

In classical physics, both the spatial position and the momentum of a particle can be determined accurately and so the state of a particle can be represented by a point in the phase-space. In Quantum Mechanics, both the spatial position ($\boldsymbol{r}$) and the momentum ($\boldsymbol{p}$) of the particle cannot be measured accurately since they are canonically conjugate variables. According to Heisenberg's uncertainty principle $\Delta x \Delta p_x \sim h$, $\Delta y \Delta p_y \sim h$, $\Delta z \Delta p_z \sim h$, where h denotes the Planck constant. So, in Quantum Physics, the state of a particle cannot be represented by a point but only by a cell of finite volume h^3 in the phase-space diagram. So, the number of states available for the particle is given by [1]

$$\text{Number of states available for the particle} = \frac{\text{Volume of phase space}}{h^3}. \qquad (1)$$

Let us now enumerate the number of states $Z(p)$ available for a particle with

momentum lying between $\boldsymbol{p}$ and $\boldsymbol{p} + d\boldsymbol{p}$, confined within a volume V.

$$Z(p)dp = \frac{V4\pi p^2 dp}{h^3}. \tag{2}$$

From this, we obtain the density of states available for a particle in the momentum range $\boldsymbol{p}$ and $\boldsymbol{p} + d\boldsymbol{p}$.

$$z(p)dp = \frac{Z(p)dp}{V} = \frac{4\pi p^2 dp}{h^3}. \tag{3}$$

If we invoke the particle-wave duality, engrained in Quantum Mechanics, the density of states can be expressed in terms of wavelength λ or wave number k or frequency ν. If λ is the de Broglie wavelength, then $p = \dfrac{h}{\lambda}$ and $dp = -\dfrac{h}{\lambda^2}d\lambda$. Substituting these in Eq. (3), we get

$$z(\lambda)d\lambda = \frac{4\pi}{h^3}\left(-\frac{h^3}{\lambda^4}d\lambda\right) = -\frac{4\pi}{\lambda^4}d\lambda.$$

The negative sign in the above equation merely indicates that the density of states decreases with increase of wavelength; whereas the density of states in terms of momentum p increases with increase of momentum. So, we can ignore the negative sign in any discussion that follows, since λ occurs in the denominator and it automatically takes care of the decrease in density of states with increase of wavelength. So, we can rewrite the above equation, omitting the negative sign.

$$z(\lambda)d\lambda = \frac{4\pi}{\lambda^4}d\lambda. \tag{4}$$

In terms of the wave number k $(k = \dfrac{2\pi}{\lambda})$, the momentum $p = \hbar k$, where $\hbar = h/(2\pi)$ and consequently, the density of states becomes

$$z(k)dk = \frac{4\pi}{h^3}\hbar^3 k^3 dk = \frac{k^3}{2\pi^2}dk. \tag{5}$$

Since $p = \dfrac{h}{\lambda} = \dfrac{h\nu}{v}$, where ν denotes the frequency and v denotes the velocity, the density of states $z(\nu)$ in terms of the frequency ν is obtained by substitution in Eq. (3).

$$z(\nu)d\nu = \frac{4\pi\nu^2}{v^3}d\nu. \tag{6}$$

For photons, $v = c$ and so the density of states for photons is

$$z(\nu)d\nu = \frac{4\pi\nu^2}{c^3}d\nu. \tag{7}$$

Equations (3) - (6) are expressions for the density of states given in different variables : momentum p, wavelength λ, wave number k and frequency ν. **All of them are equally applicable for both particles and waves since quantum mechanics emphasizes particle-wave duality.** Eq. (7) is applicable only to photons that are propagated with velocity c.

3   Planck's law of radiation

Max Planck in the year 1900 deduced the radiation law using classical physics but introducing only an additional concept that the radiation is absorbed or emitted by the walls of the black body in integral multiples of $h\nu$, where h is a constant, known as Planck's constant. He obtained an expression for the spectral distribution of energy as a function of temperature. That is known as Planck's law of black body radiation. In most of the standard textbooks, the same old derivation is given.

It is now an established knowledge that radiation or light is of quantal nature, each quantum carrying an energy $h\nu$. It is also known that photons obey Bose-Einstein statistics, from which one can find the probability of occupation of a state.

From Eq. (7), we obtain the number of states available for photons of frequency lying between ν and $\nu + d\nu$ in unit volume.

$$z(\nu)d\nu = \frac{4\pi\nu^2}{c^3}d\nu.$$

Photons are Bosons and so they should obey the Bose-Einstein statistics. The probability of occupation of a state according to Bose-Einstein statistics is

$$\frac{1}{e^{h\nu/(k_B T)} - 1}, \tag{8}$$

where k_B denotes the Boltzmann constant and T the absolute temperature. That is how S.N. Bose argued and derived Planck's law of radiation.

Since each photon can have two spin states (two transverse polarizations) for each frequency or energy state, two photons can occupy each frequency state. Taking into account the occupation probability (8) for each frequency state, we obtain the number of photons per unit volume with frequency lying between ν and $\nu + d\nu$.

$$n(\nu)d\nu = 2 \times \frac{4\pi\nu^2 d\nu}{c^3} \times \frac{1}{e^{h\nu/(k_B T)} - 1} = \frac{8\pi\nu^2}{c^3}\frac{1}{e^{h\nu/(k_B T)} - 1}d\nu. \tag{9}$$

Multiplying the number density of photons by its energy $h\nu$, we obtain Planck's formula for the energy density of photons in an enclosure in the frequency range lying between ν and $\nu + d\nu$ in thermodynamic equilibrium at temperature T.

$$u(\nu)d\nu = \frac{8\pi\nu^2}{c^3}\frac{h\nu}{e^{h\nu/(k_B T)} - 1}d\nu. \tag{10}$$

One can also write the spectral distribution in terms of angular frequency $\omega = 2\pi\nu$ or wavelength λ.

$$u(\omega)d\omega = \frac{\omega^2}{\pi^2 c^3} \frac{\hbar\omega}{e^{\hbar\omega/(k_B T)} - 1} d\omega. \tag{11}$$

$$u(\lambda)d\lambda = \frac{8\pi}{\lambda^5} \frac{hc}{e^{hc/(\lambda k_B T)} - 1} d\lambda. \tag{12}$$

In Eq. (11), the energy of the photon $h\nu$ is written as $\hbar\omega$, where $\hbar = h/(2\pi)$.
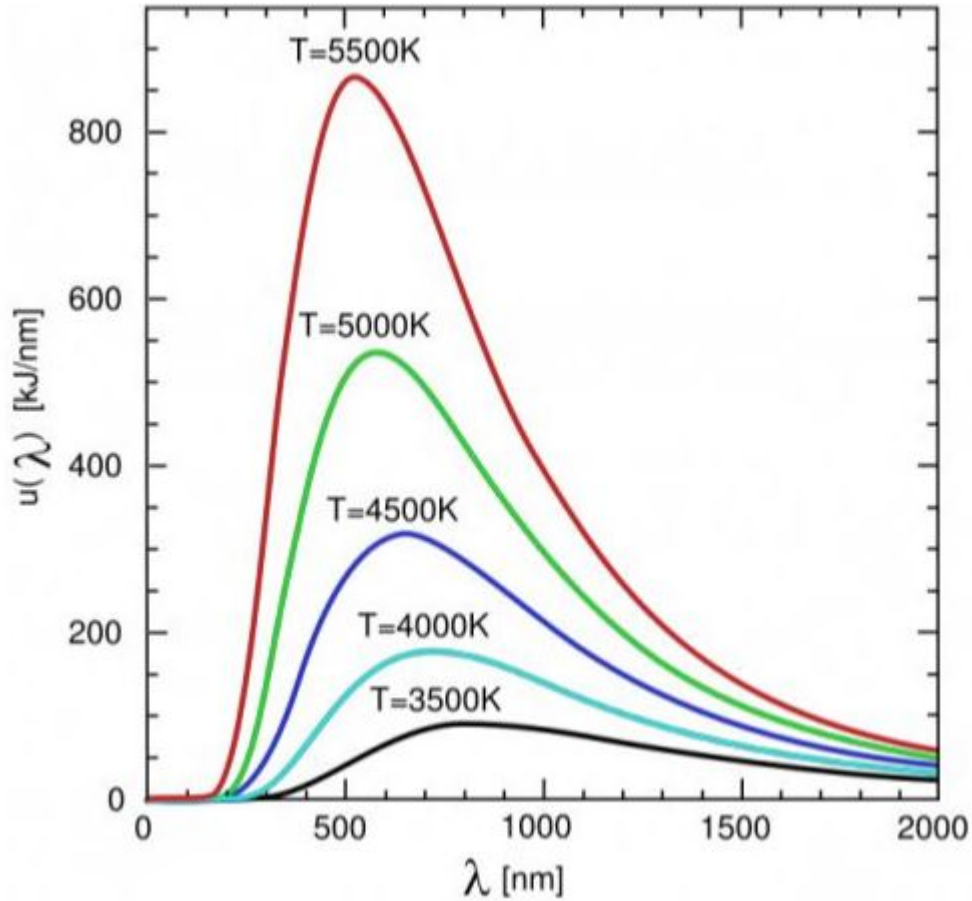

Fig 1:Variation of spectral energy density $u(\lambda)$ with wavelength λ in nano-meters. The wavelength $\lambda_{\max}$ at which $u(\lambda)$ is maximum is observed to shift towards shorter wavelength as T increases.

The variation of spectral energy distribution $u(\lambda)$ with wavelength λ is shown in Fig. 1. for various temperatures T. It is found that $u(\lambda)$ increases with λ, reaches a maximum at certain wavelength $\lambda_{\max}$ and then falls off. The higher the temperature, the higher is the corresponding $u(\lambda)$ curve but the $\lambda_{\max}$ shifts towards the shorter wavelength as T increases, according to Wein's displacement law $\lambda_{\max}T = $ constant.

For the sake of completeness, it is shown in Appendix A how Planck's formula reduces to Releigh-Jeans formula in the limit of long wavelegths and Wein's formula for short wavelengths. What is experimentally observed is the spectral radiant energy emitted per second per unit area of the black body and also the total energy emitted per second as given by Stefan's law. A discussion on these aspects is also included in the Appendix A.

4 Free electron quantum theory of metals

A metal can be considered to have a lattice structure with atoms located at the lattice points. The valence electrons are loosely bound and so they get detached and move freely throughout the metal whereas the residual positive ions are located at the lattice points.

Thus a metal is visualized to consist of positive ions located at the lattice points with the free movement of valance electrons within a volume V of the metal. Since electrons are confined within a volume V, their energy levels will be discrete.

The number of states available for the electron with momentum lying between p and $p + dp$ is given by Eq. (2).

$$Z(p)dp = \frac{V 4\pi p^2 dp}{h^3}.$$

This can be expressed in terms of the kinetic energy of electron $E = p^2/(2m)$, where m denotes the mass of the electron. Substituting $p = \sqrt{2mE}$ in the above expression, we get the number of states lying between energy E and $E + dE$.

$$Z(E)dE = \frac{4\pi V}{h^3}(2mE)^{1/2}mdE = \frac{4\sqrt{2}\,\pi V m^{3/2} E^{1/2} dE}{h^3}. \tag{13}$$

Electrons are spin-$\frac{1}{2}$ particles and so each electron can have two spin states, one with spin-up and another with spin-down. Electrons obey the Pauli-exclusion principle that no two identical particles can occupy the same state. If E_F is the maximum energy of the electron, then the total number of electrons that can be accommodated in volume V is

$$\begin{aligned} N &= 2 \times \int_0^{E_F} Z(E)dE = \frac{8\sqrt{2}\,\pi V m^{3/2}}{h^3} \int_0^{E_F} E^{1/2} dE \\ &= \left(\frac{16\sqrt{2}\,\pi V m^{3/2}}{3h^3}\right) E_F^{3/2}. \end{aligned} \tag{14}$$

In terms of the electron density $n = N/V$, the Fermi energy can be written as

$$E_F = \left(\frac{3h^3 n}{16\sqrt{2}\pi m^{3/2}}\right)^{2/3} = \frac{h^2}{2m}\left(\frac{3n}{8\pi}\right)^{2/3} \tag{15}$$

It is a common practice to use $\hbar = h/(2\pi)$ instead of h. In terms of $\hbar$, the Fermi energy can be written as

$$E_F = \frac{\hbar^2}{2m}(3\pi^2 n)^{2/3}. \tag{16}$$

If there are N electrons confined within a volume V, then the ground state of the system is one in which all the energy levels up to E_F are filled. This ground state will correspond to 0° K.

4.1   Fermi distribution function

The Fermi distribution function is given by

$$f(E,T) = \frac{1}{e^{(E-E_F)/(k_B T)} + 1}, \tag{17}$$

where E is the energy of the state, E_F the Fermi energy and k_B, the Boltzmann constant. It gives the probability of occupation of the state with energy E at absolute temperature T. For $T = 0$,

$$f(E, T = 0) = \begin{cases} 1, & \text{if } E < E_F \\ 0, & \text{if } E > E_F \end{cases}. \tag{18}$$

At $T = 0^o$ K, all the energy states below E_F are filled and all the energy states above E_F are empty. With the increase of temperature, the states with energy levels just below the Fermi level are depleted and the states immediately above the Fermi levels are occupied as shown in Fig.2.
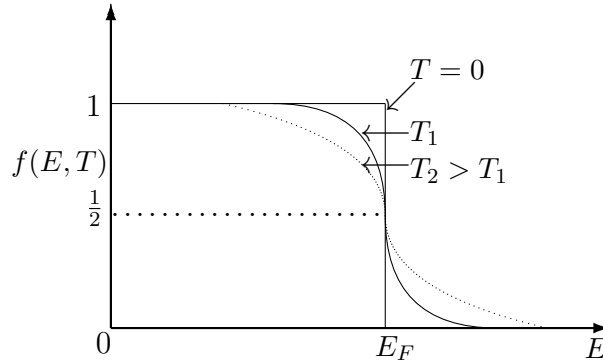

Fig. 2: Fermi distribution as a function of temperature

4.2   Fermi energy of metals

The Fermi energy depends on the electron density n, which, in turn, depends on the valence of the metal and also on the crystal structure. The Fermi energy is different

for different metals and so their properties also vary.

Let us illustrate this point by considering four mono-valent metals, Sodium, Copper, Silver and Gold, one di-valent metal Magnesium and one tri-valent metal Aluminium with the following electron configurations:

Sodium:	^{11}Na:	$1s^2\,2s^2\,2p^6\,3s$
Copper:	^{29}Cu:	$1s^2\,2s^2\,2p^6\,3s^2\,3p^6\,3d^{10}\,4s$
Silver:	^{47}Ag:	$1s^2\,2s^2\,2p^6\,3s^2\,3p^6\,3d^{10}\,4s^2\,4p^6\,4d^{10}\,5s$
Gold:	^{79}Au:	$1s^2\,2s^2\,2p^6\,3s^2\,3p^6\,3d^{10}\,4s^2\,4p^6\,4d^{10}\,5s^2\,5p^6\,5d^{10}\,5s$
Magnesium:	^{12}Mg:	$1s^2\,2s^2\,2p^6\,3s^2$
Aluminium:	^{13}Al:	$1s^2\,2s^2\,2p^6\,3s^2\,3p$

Each of the metals, Sodium, Copper, Silver and Gold has one electron in the outermost s-orbit which get detached to form the free electron cloud shared by the entire metal. In Magnesium, there are two s-electrons which are loosely bound in the outermost orbit and hence get detached to form the valence electron cloud. In Aluminium, two s-electrons and one p-electron in the outermost orbits contribute to the free valence electrons. It is possible to find the free electron density n in each of these metals from a knowledge of their crystal structure.

Sodium has a *bcc* structure, whereas Copper, Silver, Gold and Aluminium have *fcc* structure and Magnesium has *hcp* structure. From a knowledge of their lattice constants, one can determine the number of atoms in unit volume n_{atom}. Given the valency, one can find the number of electrons per unit volume ($n = \text{valency} \times n_{\text{atom}}$).

bcc solid

In a Body-Centered Cubic (*bcc*) solid, there are two atoms per unit cell of volume a^3, where a denotes the lattice constant. So, the number of atoms per unit volume is equal to $2/a^3$. That is $n_{\text{atom}} = 2/a^3$. If we multiply n_{atom} by the valency of atom, we get the number of free electrons per unit volume: $n = \text{valency} \times n_{\text{atom}}$.

$$\text{For } bcc \text{ solid:} \quad n_{\text{atom}} = \frac{2}{a^3}; \quad n = \text{valency} \times n_{\text{atom}}. \tag{19}$$

fcc solid

In a Face-Centered Cubic (*fcc*) solid, there are four atoms per unit cell of volume a^3, where a denotes the lattice constant. So, the number of atoms per unit volume is equal to $4/a^3$. That is $n_{\text{atom}} = 4/a^3$. If we multiply n_{atom} by the valency of atom, we get the number of free electrons per unit volume: $n = \text{valency} \times n_{\text{atom}}$.

$$\text{For } fcc \text{ solid:} \quad n_{\text{atom}} = \frac{4}{a^3}; \quad n = \text{valency} \times n_{\text{atom}}. \tag{20}$$

hcp solid

In a Hexagonal Close Packed (*hcp*) structure, the solid has six atoms in a unit cell of volume $\dfrac{3\sqrt{3}}{2}a^2c$ where a and c are two lattice constants defining the *hcp* structure. So, the number of atoms per unit volume is equal to $\dfrac{4}{\sqrt{3}a^2c}$. That is $n_{\text{atom}} = \dfrac{4}{\sqrt{3}a^2c}$. If we multiple n_{atom} by the valency of atom, we get the number of free electrons per unit volume: $n = \text{valency} \times n_{\text{atom}}$.

$$\text{For } bcc \text{ solid:} \quad n_{\text{atom}} = \frac{4}{\sqrt{3}a^2c}; \quad n = \text{valency} \times n_{\text{atom}}. \tag{21}$$

Given the lattice structure, lattice constants and valency of the metals, one can determine the density of atoms (n_{atom}) and the density of valence free electrons (n) using Eqs. (19) - (21) and they are given in a tabular column (Table 1).

Table 1: Lattice structure, lattice constants, atomic density, valency, valence electron density and Fermi energy of some selected metals

Metal	Crystal structure	Lattice constant in n.m.	Atomic density per m^3	Valency	Valence Electron density per m^3	Fermi energy in eV
Sodium	*bcc*	$a = 0.4281$	2.65×10^{28}	1	2.65×10^{28}	3.23
Copper	*fcc*	$a = 0.3615$	8.50×10^{28}	1	8.50×10^{28}	7.05
Silver	*fcc*	$a = 0.4086$	5.85×10^{28}	1	5.85×10^{28}	5.50
Gold	*fcc*	$a = 0.4080$	5.89×10^{28}	1	5.89×10^{28}	5.52
Magnesium	*hcp*	$\begin{cases} a = 0.3209 \\ c = 0.5210 \end{cases}$	4.30×10^{28}	2	8.60×10^{28}	7.12
Aluminium	*fcc*	$a = 0.4049$	6.02×10^{28}	3	18.06×10^{28}	11.63

Our aim is to find the Fermi energy of valence electrons in any metal. The Fermi energy E_F depends on the free electron density n, which, in turn, depends on the crystal structure and the valency of the metal. From Eq.(16), we have

$$E_F = \frac{\hbar^2}{2m}(3\pi^2 n)^{2/3} = \left(\frac{\hbar^2}{2m}(3\pi^2)^{2/3} \right) n^{2/3}. \tag{22}$$

In the above Equation, we have separated the constant part from the only variable n which depends on the metal. Substituting the values

$$\hbar = \frac{h}{2\pi} = 1.055 \times 10^{-34} \text{ Js}$$
$$\text{Electron mass } m = 9.109 \times 10^{-31} \text{ Kg}$$

we get

$$E_F = 58.473 \times 10^{-39} \, n^{2/3} \text{ Joule.}$$

This can be expressed in eV, using the conversion factor 1 eV= 1.602×10^{-19} J.

$$E_F = 36.50 \times 10^{-20} \, n^{2/3} \text{ eV,} \tag{23}$$

where n represents the number of valence electrons per cubic meter. The results are tabulated in Table 1.

5 Debye's theory of specific heat of solids

Einstein considered the solid to consist of molecules at lattice points which execute simple harmonic vibrations about their equilibrium positions. This will set up standing waves in solid. So, Debye considered the solid as a continuous medium wherein standing waves of different modes are set up due to molecular vibrations. Since particles and waves are treated alike in the phase-space concept, the number of modes of standing waves can be obtained from Eq. (5) in terms of the wave number k or equivalently by Eq. (6) in terms of frequency ν and velocity v of wave propagation. If V is the volume of the solid, then the number of modes of standing waves in terms of frequency ν and velocity v of wave propagation, according to Eq. (6), is

$$Z(\nu)d\nu = \frac{V 4\pi\nu^2}{v^3}d\nu. \tag{24}$$

Since both longitudinal and transverse waves are possible in a solid, we should take into account one mode for longitudinal wave and two modes for transverse waves. It is also possible that the longitudinal and transverse waves may be propagated with different velocities v_l and v_t. Taking these factors into account, we get

$$Z(\nu)d\nu = 4\pi V \left(\frac{1}{v_l^3} + \frac{2}{v_t^3} \right) \nu^2 d\nu. \tag{25}$$

The quantum analogue of such a wave is known as phonon and so these waves are often referred to as phonons in recent literature.

5.1 Debye's postulate

If the solid contains N atoms, then $3N$ modes of linear vibrations are possible. Debye made a postulate that the total number of modes of standing waves in the solid should be equal to the $3N$ modes of vibrations that are possible in the solid.

Let the upper limit of the frequency of the wave modes be ν_D, the Debye frequency. It follows that

$$\int_0^{\nu_D} Z(\nu)d\nu \;=\; 4\pi V \left(\frac{1}{v_l^3} + \frac{2}{v_t^3}\right) \int_0^{\nu_D} \nu^2 d\nu = 3N. \tag{26}$$

This yields the Debye frequency ν_D.

$$\nu_D^3 \;=\; \frac{9N}{4\pi V}\left(\frac{1}{v_l^3} + \frac{2}{v_t^3}\right)^{-1}.$$

$$\nu_D \;=\; \left(\frac{9N}{4\pi V}\right)^{1/3}\left(\frac{1}{v_l^3} + \frac{2}{v_t^3}\right)^{-1/3}. \tag{27}$$

One can obtain a rough estimate of the Debye frequency. Assuming that $N/V = 10^{28}$ per m^3 and the velocity of propagation of waves is approximately equal to that of sound waves $v_l = v_t = 1000$ m/sec, we obtain

$$\nu_D = \left(\frac{9 \times 10^{28}}{4\pi}\right)^{1/3}\left(\frac{3}{1000^3}\right)^{-1/3} \approx 10^{12} Hz.$$

5.2   Calculation of specific heat

Let us associate each mode of wave with a corresponding harmonic oscillator of the same frequency. It can be shown that the mean energy of a linear harmonic oscillator with frequency ν is

$$\bar{E} = \frac{h\nu}{e^{h\nu/(k_B T)} - 1}.$$

Associating each wave mode given by Eq. (25) with the corresponding energy of the linear harmonic oscillator, the total vibrational energy of the solid is obtained by integrating over the frequency range from zero to ν_D.

$$\begin{aligned} E &= \int_0^{\nu_D} Z(\nu)\left(\frac{h\nu}{e^{h\nu/(k_B T)} - 1}\right) d\nu \\ &= 4\pi V \left(\frac{1}{v_l^3} + \frac{2}{v_t^3}\right)\int_0^{\nu_D}\left(\frac{h\nu^3}{e^{h\nu/(k_B T)} - 1}\right) d\nu. \end{aligned} \tag{28}$$

From Eq. (27), we find

$$4\pi V \left(\frac{1}{v_l^3} + \frac{2}{v_t^3}\right) = \frac{9N}{\nu_D^3}.$$

Let us also make the following substitutions:

$$x = \frac{h\nu}{k_B T}; \quad x_m = \frac{h\nu_D}{k_B T}.$$

Then, Eq. (28) becomes

$$E = 9Nk_BT \left(\frac{k_BT}{h\nu_D}\right)^3 \int_0^{x_m} \frac{x^3}{e^x - 1} dx. \tag{29}$$

The specific heat of solid C_V is given by $C_V = \frac{dE}{dT}$. One can obtain analytical expressions for C_V in the limiting cases of large T and small T.

It is useful to define Debey temperature θ_D by the relation

$$k_B\theta_D = h\nu_D, \tag{30}$$

such that

$$\theta_D = \frac{h\nu_D}{k_B} \quad \text{and} \quad x_m = \frac{h\nu_D}{k_BT} = \frac{\theta_D}{T}. \tag{31}$$

5.2.1 At very high temperatures $T \gg \theta_D$

Since $x = \frac{h\nu}{k_BT} \ll 1$, $e^x \approx 1 + x$. Eq. (29) reduces to

$$\begin{aligned} E &= 9Nk_BT \left(\frac{k_BT}{h\nu_D}\right)^3 \int_0^{x_m} x^2 dx \\ &= 9Nk_BT \left(\frac{T}{\theta_D}\right)^3 \frac{x_m^3}{3} \\ &= 3Nk_BT. \\ C_V = \frac{dE}{dT} &= 3Nk_B. \end{aligned} \tag{32}$$

Equation (32) is obtained by substituting the value of x_m from Eq. (31). Thus, at very high temperatures, we get the classical result of $3Nk_B$ for the specific heat of solid $C_v = \frac{dE}{dT}$.

5.2.2 At very low temperatures $T \ll \theta_D$

At very low temperatures, the upper limit of integration $x_m \to \infty$. Using the value of the standard integtral,

$$\int_0^\infty \frac{x^3}{e^x - 1} dx = \frac{\pi^4}{15},$$

in Eq. (29), we get

$$\begin{aligned} E &= 9Nk_BT \left(\frac{k_BT}{h\nu_D}\right)^3 \left(\frac{\pi^4}{15}\right) \\ &= \frac{3}{5}\pi^4 Nk_B \frac{T^4}{\theta_D^3}. \end{aligned} \tag{33}$$

Differentiating (33) with respect to temperature T, we obtain the specific heat of solid at very low temperature.

$$C_V = \frac{dE}{dT} = \left(\frac{12}{5}\right)\pi^4 N k_B \left(\frac{T}{\theta_D}\right)^3. \tag{34}$$

This is the famous Debye's T^3 law which is found to yield excellent agreement with experimental results for many solids at very low temperatues.

6   The Fermi Gas Model of the Nucleus

Here, we consider the nucleus as a sphere of volume V consisting of A non-interacting nucleons. Nucleons are fermions and they obey Pauli exclusion principle. No two identical fermions can occupy the same state. The number of states available for nucleons with momentum lying between p and $p = dp$ is given by Eq. (2) from phase-space consideration [1].

$$Z(p)dp = \frac{V 4\pi p^2 dp}{h^3}. \tag{35}$$

Each state can be occupied by 4 nucleons, proton with spin up and spin down and neutron with spin up and down. So, the total number of nucleons that can be accommodated in a nucleus of volume V with Fermi momentum p_F (the maximum momentum) is

$$A = \frac{4V}{h^3}\int_0^{p_F} 4\pi p^2 dp = \frac{4V}{h^3}\left(\frac{4\pi p_F^3}{3}\right). \tag{36}$$

This can be expressed in terms of Fermi energy $E_F = p_F^2/(2M)$, where M denotes the nucleon mass. This yields the nuclear density (number of nucleons per unit volume of the nucleus) $n = A/V$.

$$n = \frac{A}{V} = \frac{16\pi p_F^3}{3h^3} = \frac{32\sqrt{2}\pi M^{3/2}}{3h^3}E_F^{3/2}. \tag{37}$$

In a nucleus, the nuclear density is a constant and so, the Fermi energy is the same for all nuclei.

The Fermi gas model for the nuclei is of limited validity, since the number of nucleons even in heavy nucleus is very small and is of the order of 200 only. Besides the nucleons within the nucleus are tightly bound with a binding energy of the order of MeV. In spite of the limited validity, the Fermi gas model for the nucleus is used to find some of the bulk properties of nuclei [1].

Acknowledgement

Appendix A

It can be shown that Planck's radiation law reduces to Raleigh-Jeans law at high temperatures and Wein's law at low temperatures. From Wein's law, one can obtain Wein's displacement law. One can also obtain the total energy density by integrating the spectral energy density $u(\nu)$ over all frequencies.

What is experimentally observed is the spectral distribution of energy emitted in the black body radiation and the total energy emitted. This leads to Stefan's law and the determination of Stefan's constant.

Raleigh-Jeans formula

In the limit of long wavelegths, the exponential function can be approximated.

$$e^{hc/(\lambda k_B T)} \longrightarrow 1 + \frac{hc}{\lambda k_B T}.$$

Substituting this in Eq: (12), we obtain the Raleigh-Jeans formula

$$u(\lambda) = \frac{8\pi k_B T}{\lambda^4}. \tag{A.1}$$

Wein's formula

In the limit of short wavelengths,

$$e^{hc/(\lambda k_B T)} \gg 1.$$

Hence

$$e^{hc/(\lambda k_B T)} - 1 \approx e^{hc/(\lambda k_B T)}.$$

Substituting this in Eq. (12), we obtain Wein's formula.

$$u(\lambda) = \frac{8\pi hc}{\lambda^5} e^{-hc/(\lambda k_B T)} = \frac{A}{\lambda^5} e^{-B/(\lambda T)}, \tag{A.2}$$

where A and B are two constants.

$$A = 8\pi hc; \qquad B = \frac{hc}{k_B}. \tag{A.3}$$

Wein's displacement law

Given Eq. (A.2), one can find the wavelength at which $u(\lambda)$ is maximum, using the condition that $\frac{du_\lambda}{d\lambda} = 0$.

$$\frac{du(\lambda)}{d\lambda} = A\left(-\frac{5}{\lambda^6}\right)e^{-B/(\lambda T)} + \frac{A}{\lambda^5}\frac{d}{d\lambda}\left(e^{-B/\lambda T}\right) = 0. \tag{A.4}$$

Since

$$\frac{d}{d\lambda}\left(e^{-B/(\lambda T)}\right) = \frac{B}{T\lambda^2}e^{-B/(\lambda T)},$$

we get the condition for the wavelength λ_{max}, at which the energy density is maximum.

$$\lambda_{max}T = \frac{B}{5} = \frac{hc}{5k_B} = \text{constant}. \tag{A.5}$$

Substituting the values $h = 6.625 \times 10^{-34}$ J sec, $c = 2.998 \times 10^8$ m/sec, $k_B = 1.38 \times 10^{-23}$ J/K, we can determine the constant.

$$\lambda_{max}T = \frac{hc}{5k_B} = 2.88 \times 10^{-3} \text{ m.K.} \tag{A.6}$$

According to Eqs. (A.5) and (A.6), $\lambda_{max}T$ is a constant and this is known as Wein's displacement law.

Total energy density integrated over all frequencies

If the spectral energy density $u(\nu)$ given by Eq. (10) is integrated over all the frequencies, we get the total energy density in the cavity.

$$U = \int u(\nu)d\nu = \frac{8\pi h}{c^3}\int_0^\infty \frac{\nu^3}{e^{h\nu/(k_B T)} - 1}d\nu. \tag{A.7}$$

Let us put $y = h\nu/(k_B T)$. Then $dy = h/(k_B T)d\nu$. Now the integral becomes

$$U = \frac{8\pi h}{c^3}\left(\frac{k_B T}{h}\right)^4\int_0^\infty \frac{y^3}{e^y - 1}dy. \tag{A.8}$$

Substituting the value of the standard integral

$$\int_0^\infty \frac{y^3}{e^y - 1}dy = \frac{\pi^4}{15},$$

we get

$$U = \frac{8\pi^5 k_B^4}{15c^3 h^3}T^4 = \alpha T^4, \tag{A.9}$$

where $\alpha = \dfrac{8\pi^5 k_B^4}{15c^3h^3}$. Substituting the values of k_B, c, h, we get

$$\alpha = 7.55 \times 10^{-16}\ \mathrm{Jm^{-3}K^{-4}}.$$

Thus, we find that the total energy density integrated over all the wavelengths is proportional to the fourth power of the absolute temperature.

Energy emitted per second per unit area of black body

We have calculated the spectral energy density of waves $u(\nu)$ in the cavity. Then the spectral radiant energy emitted per second per unit area of the black body is $cu(\nu)$, where c is the velocity of light. This is emitted uniformly in all directions. If E_ν is the spectral energy emitted in frequency ν per unit solid angle, then

$$4\pi E_\nu = cu(\nu) \quad \text{or} \quad E_\nu = \frac{cu(\nu)}{4\pi}. \tag{A.10}$$

What is experimentally observable is the spectral radiant energy emitted by the black body. So, let us write down Planck's expression for the spectral radiant energy emitted per second per unit solid angle by unit area of the black body .

$$
\begin{aligned}
E_\nu d\nu &= \frac{cu(\nu)}{4\pi} d\nu = \frac{2\nu^2}{c^2} \frac{h\nu}{e^{h\nu/(k_B T)} - 1} d\nu. & \text{(A.11)} \\
E_\omega d\omega &= \frac{cu(\omega)}{4\pi} d\omega = \frac{\omega^2}{4\pi^3 c^2} \frac{\hbar\omega}{e^{\hbar\omega/(k_B T)} - 1} d\omega. & \text{(A.12)} \\
E_\lambda d\lambda &= \frac{cu(\lambda)}{4\pi} d\lambda = \frac{2c}{\lambda^5} \frac{hc}{e^{hc/(\lambda k_B T)} - 1} d\lambda. & \text{(A.13)}
\end{aligned}
$$

Total radiant spectral energy emitted per second per unit area in the forward direction

If E_ν is the radiant energy emitted per unit area normal to the surface per second, then the total energy $\mathcal{E}_\nu$ emitted per unit area per sec in the forward direction is

$$
\begin{aligned}
\mathcal{E}_\nu &= \int E_\nu \cos\theta\, d\Omega \\
&= 2\pi \int_0^{\pi/2} E_\nu \cos\theta \sin\theta\, d\theta \\
&= 2\pi E_\nu \int_0^1 x\, dx, \quad \text{where } x = \sin\theta \\
&= \pi E_\nu = \frac{cu(\nu)}{4}. \tag{A.14}
\end{aligned}
$$

Equation (A.14) is obtained by substituting (A.10) for E_ν.

Stefan-Boltzmann law

The total energy of all wavelengths radiated per second per unit area by a black body at absolute temperature T is given by Stefan-Boltzmann law.

$$\mathcal{E} = \sigma T^4, \tag{A.15}$$

where σ is known as the Stefan constant and it is found experimentally to yield

$$\sigma = 5.67 \times 10^{-8} \text{ Watts m}^{-2}\text{K}^{-4}.$$

Let us now deduce Stefan's law from Planck's radiation formula and calculate the Stefan constant.

We have already derived an expression (A.9) for the total energy density U in the cavity of the black body and we have found it proportional to T^4. The total spectral radiant energy from the unit area of the black body is given by Eq. (A.14). Integrating over all the frequencies, we obtain the total energy radiated by unit area of the black body.

$$\mathcal{E} = \int_0^\infty \mathcal{E}_\nu d\nu = \frac{c}{4} \int_0^\infty u(\nu) d\nu = \frac{c}{4} U = \frac{c}{4} \alpha T^4. \tag{A.16}$$

Comparing (A.15) and (A.16), we get

$$\sigma = \frac{c}{4} \alpha = 5.66 \times 10^{-8} \text{ Watts m}^{-2}\text{K}^{-4}. \tag{A.17}$$

This agrees well with the experimental value.

References

1. V. Devanathan, *Nuclear Physics, sec. 2.1*, Narosa Publishing House, New Delhi (2011); Alpha Science International, U.K. (2011)

2. W. Greiner, *Quantum Mechanics-An Introduction*, Springer-Verlag, Berlin (1989)

3. F. Schwabl, *Quantum Mechanics*, Springer-Verlag, Berlin (1992)

4. C. Kittel & H. Kroemer, *Thermal Physics*, W.H. Freeman & Company, New York (1980)

5. L. N. Cooper, *An Introduction to the Meaning and Structure of Physics*, Harper & Row, New York (1968)

6. S. O. Pillai, *Solid State Physics, 8th Edition*, New Age International, New Delhi (2018)

7. R. Asokamani, *Solid State Physics, 2nd Edition*, Eswar Press, Chennai (2015)

8. P. K. Palanisamy, *Material Science*, Scitech (2003)

9. S. L. Kakani & H. M. Chandalia, *Quantum Mechanics, 4th Edition*, Sultan Chand, New Delhi (2004)